Math 132: Differential Topology

§ Whitney embedding theorem

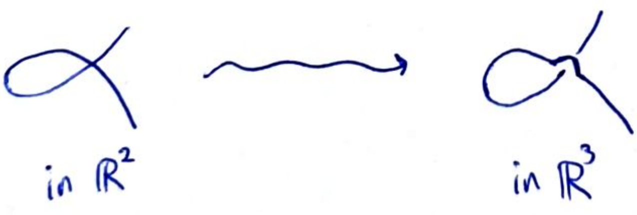
So far, our m-manifold M has been embedded in $\mathbb{R}^k$ where k maybe enormous. Can we embed it in a smaller Euclidean space?

Thm (strong Whitney embedding theorem)
Any smooth m-manifold (with $m>0$) can be embedded in $\mathbb{R}^{2m}$.

In this lecture, we'll only prove a weaker version where we embed it in $\mathbb{R}^{2m+1}$; going further down to $\mathbb{R}^{2m}$ requires a method known as the "Whitney trick".

Intuitively, it's clear why it should be possible to embed M^m into $\mathbb{R}^{2m+1}$: since $m+m < 2m+1$, there's enough room to wiggle thing around,

e.g.



in $\mathbb{R}^2$ in $\mathbb{R}^3$

Rmk Whitney's result is the best possible linear bound.
In fact, when m is a power of 2, there's no embedding $\mathbb{RP}^m \hookrightarrow \mathbb{R}^{2m-1}$.

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Now, let's prove

<u>Thm</u> Every m-manifold admits an injective immersion in $\mathbb{R}^{2m+1}$.

proof) If $M^m \subset \mathbb{R}^k$, ~~and~~ we will produce a linear projection $\mathbb{R}^k \to \mathbb{R}^{2m+1}$ that restricts to an injective immersion of M. Proceeding inductively, it suffices to show that: if $f: M \to \mathbb{R}^\ell$ is an injective immersion with $\ell > 2m+1$, then there exists a unit vector $a \in \mathbb{R}^\ell$ such that the composition of f with the projection map $\pi_a : \mathbb{R}^\ell \longrightarrow H_a := \{b \in \mathbb{R}^\ell \mid b \perp a\}$ is still an injective immersion.

Define maps
$$h : M \times M \times \mathbb{R} \longrightarrow \mathbb{R}^\ell$$
$$(x, y, t) \mapsto t(f(x) - f(y))$$

and
$$g : TM \longrightarrow \mathbb{R}^\ell$$
$$(x, v) \longmapsto df_x(v).$$

Since $\ell > 2m+1$, by <u>Sard</u>, there exist a point $a \in \mathbb{R}^\ell \setminus \{0\}$ belonging to neither image.

Then $\pi_a \circ f : M \to H_a$ is injective, since

$\pi_a \circ f(x) = \pi_a \circ f(y) \Rightarrow f(x) - f(y) = ta$ for some $t \in \mathbb{R}$ $\Rightarrow$ either $t = 0$ (which means $x = y$ by injectivity of f) or $h(x, y, \frac{1}{t}) = a$ (contradiction to our choice of a)

Moreover, $\pi_a \circ f$ is an immersion, since

$d(\pi_a \circ f)_x(v) = 0 \Rightarrow \pi_a \circ df_x(v) = 0 \Rightarrow df_x(v) = ta$ for some $t \neq 0$ by immersivity of f

$\Rightarrow g(x, \frac{1}{t}) = a$, a contradiction. ∎

For compact manifolds, injective immersions are the same as embeddings, so we have just proved the (weaker version of) embedding theorem in the compact case.

For non compact manifolds, we need to modify the immersion to make it proper, which is a topological, not a differential problem.

The fundamental technique for such generalization is the partion of unity.

Def Let M be a smooth manifold.

A partition of unity $\{\rho_i\}_{i\in I}$ is a set of smooth functions $\rho_i : M \to \mathbb{R}$ such that

(a) $\rho_i \geq 0$

(b) Each $x \in M$ has a neighborhood on which all but finitely many ρ_i are identically zero

(c) $\sum_{i\in I} \rho_i(x) = 1$ for all $x \in M$.

If $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M, then $\{\rho_i\}_{i\in I}$ is subordinate to $\{U_\alpha\}_{\alpha\in A}$ if each $\operatorname{supp}\rho_i = \overline{\rho^{-1}(\mathbb{R}_{\neq 0})}$ is contained in some U_α.

Thm Every open cover $\{U_\alpha\}_{\alpha\in A}$ admits a countable partition of unity $\{\rho_i\}_{i\in I}$ subordinate to it.

(For now, let's take this theorem for granted.)

<u>Cor</u> On any manifold M, there exists a proper function $f: M \to \mathbb{R}$.

proof) Let $\{U_\alpha\}_{\alpha \in A}$ be the set of open subsets of M with compact closure.

Choose a partition of unity $\{\rho_i\}_{i \in \mathbb{Z}_{>0}}$ subordinate to it.

Define $f = \sum_{i=1}^{\infty} i \rho_i$. Then $f^{-1}([-j, j]) \subset \bigcup_{i=1}^{j} \operatorname{supp} \rho_i$, so f is proper. ∎

Now we're ready to prove the

<u>Thm</u> Every m-manifold embeds in $\mathbb{R}^{2m+1}$.

proof) Begin with an injective immersion $M \longrightarrow \mathbb{R}^{2m+1}$.

Composing with a diffeomorphism of $\mathbb{R}^{2m+1}$ into its unit ball, $z \longmapsto \frac{z}{1+|z|^2}$,

we obtain an injective immersion $f: M \to \mathbb{R}^{2m+1}$ with $|f(x)| < 1$ for all $x \in M$.

Let $g: M \to \mathbb{R}$ be a proper function, and define a new injective immersion

$$F: M \to \mathbb{R}^{2m+2}$$

$$x \longmapsto (f(x), g(x)).$$

Recall that $\pi_a \circ F: M \longrightarrow H_a \cong \mathbb{R}^{2m+1}$ is an injective immersion for almost every $a \in S^{2m+1}$,

so pick an a that is neither of the sphere's two poles.

We claim that $\pi_a \circ F$ is proper. In fact, we will show that, for any c,

there exists d such that $(\pi_a \circ F)^{-1}(\overline{B}_c) \subset g^{-1}(\overline{B}_d)$, which is compact since g is proper.

If the claim is false, there should be a sequence of points $\{x_i\}$ in M

for which $|\pi_a \circ F(x_i)| < c$ but $g(x_i) \longrightarrow \infty$.

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By the definition of π_a,

$$w_i := \frac{1}{g(x_i)}\left(F(x_i) - \pi_a \circ F(x_i)\right) \in \mathbb{R}^{2m+2}$$

is a multiple of a.

As $i \to \infty$,

$$w_i = \frac{F(x_i)}{g(x_i)} - \frac{\pi_a \circ F(x_i)}{g(x_i)} = \left(\overbrace{\frac{f(x_i)}{g(x_i)}}^{\text{has norm} \leq 1}, 1\right) - \frac{\overbrace{\pi_a \circ F(x_i)}^{\text{has norm} \leq c}}{\underbrace{g(x_i)}_{\to\infty}}$$

$$\xrightarrow{i \to \infty} (0, \dots, 0, 1).$$

Since each w_i is a multiple of a, so should be the limit.

This contradicts our choice of a, and therefore $\pi_a \circ F$ is proper. ∎